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ON TRANSMISSION OF SOUND IN A NON-UNIFORM DUCT CARRYING A SUBSONIC COMPRESSIBLE FLOW

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The differential equations governing the transmission of one-dimensional sound waves in a non-uniform duct carrying a subsonic compressible mean flow have been the subject of a recent debate [1, 2]. Of the two formulations presented, one is considered to be non-acoustical and the other as neglecting the spatial variation of the speed of sound. The present paper shows that both formulations are acoustical and represent valid approximations to correct conditions for isentropic sound propagation in a subsonic low Mach number duct. Each formulation is associated with an "error wave", which is essentially a hydrodynamic wave when the mean flow Mach number is small. Three-port modelling is required, however, to capture this wave when the Mach number of the mean flow is relatively large and a numerical matrizant approach is described which can be used for this purpose.

1. INTRODUCTION

As has been noted in a recent debate between Gupta [1] and Miles [2], some confusion appears to exist on the differential equations governing the one-dimensional sound propagation in a non-uniform duct carrying a compressible subsonic mean flow. The debate is related to an earlier paper by Gupta, Easwaran and Munjal [3], which presented a numerical solution of the problem by using a geometrical segmentation approach. A slight discrepancy was observed between the results of the segmentation method and the corresponding results of Miles [4] which are based on the numerical evaluation of the matrizant of the governing differential equations (see equations (9)–(11)). In reference [3] this discrepancy was attributed to the effects of the higher order modes which had been neglected. Gupta [1] pointed out that this conclusion was erroneous because the higher order modes had also been neglected in the formulation of Miles [4]. He then showed that, in the limit of infinite number of segments, the segmentation approach of reference [3] becomes equivalent to the differential equation

$$\begin{bmatrix} p' \\ v' \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} \begin{bmatrix} p \\ v \end{bmatrix}, \tag{1}$$

where

$$c_{11} = [-\mathrm{i}k_0 + M_0(1 - \gamma_0 M_0^2)(\ln S)'/(1 - M_0^2)]M_0/(1 - M_0^2), \tag{2}$$

$$c_{12} = [\mathrm{i}k_0 + 2M_0(\ln S)'/(1 - M_0^2)]\rho_0 c_0/(1 - M_0^2), \tag{3}$$

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$$c_{21} = [\mathrm{i}k_0 + (\gamma_0 - 1)M_0^3(\ln S)'/(1 - M_0^2)]/(1 - M_0^2)\rho_0 c_0, \tag{4}$$

$$c_{22} = [-\mathrm{i}k_0 - (1 + M_0^2)(\ln S)'/M_0(1 - M_0^2)]M_0/(1 - M_0^2). \tag{5}$$

Here p is the acoustic pressure, v is the acoustic particle velocity, S is the pipe cross-sectional area, ρ_0 is the ambient density, c_0 is the local speed of sound, $k_0 = \omega/c_0$ is the local wavenumber and $M_0 = v_0/c_0$ is the Mach number of the mean flow, $c_0 = \sqrt{(\gamma_0 p_0/\rho_0)}$ is the speed of sound, v_0 is the mean flow velocity, p_0 is the ambient pressure, $\gamma_0 = c_{p0}/c_{v0}$ is the ratio of specific heat coefficients at constant pressure, c_{p0} and at constant volume, c_{v0}, a prime ($'$) denotes differentiation with respect to x, the pipe axis, ω, is the radian frequency and $\exp(-\mathrm{i}\omega t)$ time dependence is assumed for all acoustic quantities.

Upon observing that the elements c_{ij} of the system matrix do not agree with those given by Miles [4], Gupta [1] claimed that this was because Miles [4] had implicitly assumed in his formulation of the problem that the speed of sound is spatially constant; that is, $c_0' = 0$. Miles [2] responded by stating that the discrepancy was partly due to typographical errors; however, he did not agree that his formulation implicitly assumed $c_0' = 0$. Furthermore, he claimed that the formulation presented by Gupta [1] was flawed by being non-acoustic. Miles [2] also expanded the analysis of his earlier paper by giving, for both compressible and incompressible mean flow cases, the system differential equations for the velocity potential and the non-velocity potential formulations of the problem. Here, only the non-velocity potential formulation for the compressible flow case is of interest. The system differential equations given by Miles [2] for this case, which is also the case considered in the earlier paper by Miles [4], are

$$\begin{bmatrix} p' \\ v' \end{bmatrix} = \begin{bmatrix} B_{11} & c_{12} \\ B_{21} & c_{22} \end{bmatrix} \begin{bmatrix} p \\ v \end{bmatrix}, \tag{6}$$

where,

$$B_{11} = [-\mathrm{i}k_0 + M_0(\ln S)']M_0/(1 - M_0^2), \qquad B_{21} = \mathrm{i}k_0/(1 - M_0^2)\rho_0 c_0. \tag{7, 8}$$

Here, c_{12} and c_{22} are as given by equations (3) and (5). Thus, there is indeed a difference between the two approaches. According to Miles [2], this difference arises because he is doing acoustics and Gupta's [1] formulation is non-acoustic. According to Gupta [1], on the other hand, the difference arises because the formulation of Miles [4] tacitly assumes $c_0' = 0$, which miles [2] rejects strongly.

The present paper aims to pursue this debate not only because the related discussion has not been conclusive but also because the subject matter is related to certain fundamental issues on the isentropicity of sound propagation which have not been explicitly discussed in the open literature, hitherto. The present analysis will show that the formulations of Miles [4] and Gupta [1] are both acoustical and represent valid approximations to correct conditions for isentropic sound propagation in the case of a non-uniform duct carrying a compressible subsonic low Mach number mean flow. Each formulation is associated with an "error wave", which is essentially a hydrodynamic wave when the mean flow Mach number is small. Three-port modelling is required, however, to capture this wave when the Mach number of the mean flow is relatively large and a numerical matrizant approach is described which can be used for this purpose.

2. THEORETICAL DEVELOPMENT

2.1. ISENTROPIC PROPAGATION I

With a slight change of notation, the governing acoustic equations used by Miles [2, 4] are as follows. The continuity equation is

$$D_0\rho + \rho_0 v' - v_0(\ln \rho_0)'\rho - \rho_0(\ln v_0)'v = 0. \tag{9}$$

The momentum equation is

$$\rho_0(D_0 v + v_0' v) + v_0 v_0' \rho + p' = 0. \tag{10}$$

The energy equation is (a perfect ambient gas being assumed)

$$D_0 s = 0, \tag{11}$$

where D_0 denotes the time independent material derivative based on the mean flow velocity, $D_0 = -\mathrm{i}\omega + v_0\partial/\partial x$, ρ is the acoustic density and s denotes the entropy fluctuations. These equations come from the usual linearization of the basic one-dimensional isentropic gas dynamics equations. The latter also yield the classical isentropic steady mean flow equations which can be expressed as follows for the computation of the ambient gradients:

$$(\ln v_0)' = -(\ln S)'/(1 - M_0^2), \qquad (\ln p_0)' = \gamma_0 M_0^2(\ln S)'/(1 - M_0^2),$$
$$(\ln \rho_0)' = M_0^2(\ln S)'/(1 - M_0^2). \tag{12–14}$$

The gradient of the mean flow temperature then follows from the perfect gas law $p_0 = \rho_0 R T_0$, where T_0 denotes the mean temperature. It should be noted that these equations entail no assumption regarding the constancy of γ_0.

The following relationship is offered in reference [4] for the implementation of equation (11): namely,

$$s = c_{v0}(p - c_0^2\rho)/p_0. \tag{15}$$

This relationship comes from the state equation

$$\mathrm{d}s^* = (c_v^*/p^*)\,\mathrm{d}p^* - (c_p^*/\rho^*)\,\mathrm{d}\rho^*, \tag{16}$$

where a leading asterisk (*) indicates the total value of a quantity, that is, the sum of an ambient value, which is always denoted by a subscript "0" in this paper, and a fluctuating part: $\rho^* = \rho_0 + \rho$, $p^* = p_0 + p$, $s^* = s_0 + s$, $c_v^* = c_{v0} + c_v$, $c_p^* = c_{p0} + c_p$. If one assumes that the fluctuating parts of the specific heat coefficients are negligible as small quantities of the second order so that $c_v^* = c_{v0}$ and $c_p^* = c_{p0}$, and that c_{v0} and c_{p0} can be treated as constants, then equation (16) can be integrated analytically to obtain

$$s^* = c_{v0}\ln[p^*/(\rho^*)^{\gamma_0}] + C, \tag{17}$$

where C is an integration constant. This equation can be linearized as usual by treating the acoustic fluctuations as small quantities of the first order. This process gives, for the entropy fluctuations,

$$s = c_{v0}\ln[1 + (p - c_0^2\rho)/p_0]. \tag{18}$$

Therefore, equation (15) will be true, approximately, if

$$|p_0|^2 \gg |c_0^2\rho - p|^2. \tag{19}$$

If one assumes this to be valid, the energy equation, equation (11), which is tantamount to stating that the sound propagation is isentropic, gives

$$p = c_0^2 \rho. \tag{20}$$

This relationship is used by Miles [2, 4], Gupta [1] and Gupta *et al*. [3].

2.2. ISENTROPIC PROPAGATION II

A more direct expression for entropy fluctuations can be derived by writing equation (16) in the form

$$(p^*/c_v^*)Ds^* = Dp^* - c^{*2}D\rho^*, \tag{21}$$

where D denotes the time independent material derivative based on the total fluid velocity, $D = -\mathrm{i}\omega + (v_0 + v)\partial/\partial x$, and $c^{*2} = \gamma^* p^*/\rho^*$ is the total speed of sound; that is, $c^* = c_0 + c$. Upon assuming the fluctuating part of c^*, c, is small to the second order, equation (21) can be linearized as usual to obtain, upon using equations (13) and (14), the following expression for the entropy fluctuations:

$$(p^*/c_v^*)D_0 s = D_0 p - c_0^2 D_0 \rho. \tag{22}$$

Then, from the energy equation,

$$D_0 p = c_0^2 D_0 \rho. \tag{23}$$

This relationship, which is exact insofar as the fluctuations of the speed of sound due to acoustic temperature fluctuations is small to the second order, is certainly a more accurate consequence of isentropic propagation than equation (20) which requires that not only the fluctuating parts of the specific heat cofficients are negligible but also that their mean values are constant and the the inequality (19) is valid.

Equations (20) and (23) will be consistent if the speed of sound is spatially constant; that is, $c_0' = 0$. If the speed of sound is not constant and equation (20) is to be adopted as a necessary condition for isentropic propagation, then the more accurate condition for isentropic propagation, equation (23), can also be satisfied if one takes $p' = c_0^2 \rho'$, but, obviously, this latter relationship is mathematically inconsistent with equation (20). This inconsistency is the central point in the debate between Gupta [1] and Miles [2]. Miles [2, 4] does use equation (20) with the ‘‘pseudo-derivative’’ $p' = c_0^2 \rho'$. He shows that [2], if equation (20) is used with its mathematically proper derivative, that is, $\rho' = (p/c_0^2)'$ or $p' = (c_0^2 \rho)'$, then his formulation, equation (6), will become identical to Gupta's formulation, equation (1), but the former formulation is considered as ‘‘doing acoustics’’ and the latter is rejected as a non-acoustical one. From Gupta's point of view, on the other hand, the ‘‘pseudo-derivative’’ $p' = c_0^2 \rho'$ can be true mathematically if $c_0' = 0$ and hence his claim that Miles [2, 4] implicitly assumes that the speed of sound is spatially constant. This conclusion is, of course mathematically true but, as will be shown in the following, it is also true that Miles' formulation does provide an approximate implementation of equation (23) for the problem under consideration, if the mean flow Mach number is low enough. Here, it is perhaps appropriate to note that, in references [2, 4] there is no reference to equation (23) as the condition for isentropic propagation and no attempt is made to justify the use of the ‘‘pseudo-derivative’’ $p' = c_0^2 \rho'$ with equation (20). Therefore, the premises on which Miles' formulation is justified in this paper represents only the present author's own views.

Miles' formulation constitutes an approximate implementation of equation (23) as the isentropicity condition. The nature of this approximation can be analyzed by using equation (23) in full with equations (9) and (10) to form a coupled system of differential equations for p, v and ρ. As can be shown, after some algebra, this system of equations can be expressed in a state-space form as

$$\begin{bmatrix} p' \\ v' \\ \varepsilon' \end{bmatrix} = \begin{bmatrix} B_{11} & c_{12} & a_{13} \\ B_{21} & c_{22} & 0 \\ a_{31} & 0 & a_{33} \end{bmatrix} \begin{bmatrix} p \\ v \\ \varepsilon \end{bmatrix}, \tag{24}$$

where the acoustic variable ε is defined by

$$\varepsilon = c_0^2 \rho - p. \tag{25}$$

It is seen that the elements of the first 2×2 block of the state-space matrix in equation (24) are the same as the elements in the formulation of Miles [2], equation (6). The remaining elements of the state-space matrix are given by

$$a_{13} = M_0^2 (\ln S)'/(1 - M_0^2) \tag{26}$$

$$a_{31} = (\ln \gamma_0)' + (\gamma_0 - 1) M_0^2 (\ln S)'/(1 - M_0^2), \qquad a_{33} = \mathrm{i}k/M_0 + a_{31}. \tag{27, 28}$$

The variable ε provides a measure of the accuracy of the approximate isentropic relationship $p = c_0^2 \rho$ in the context of this theory: that is, equation (24). Equations (26) and (27) can also be expressed as $a_{13} = (\ln \rho_0)'$ and $a_{31} = (\ln c_0^2)'$, respectively. In the case of a subsonic low Mach number duct, these off-diagonal gradients have negligible effect on the solution of equation (24) (see the Appendix). Hence, upon assuming that the mean flow Mach number is small, say, $M_0 < 0{\cdot}4$, these terms can be neglected and equation (24) can be expressed as a decoupled set of equations consisting of equation (6) and the following first order differential equation for ε: namely,

$$\varepsilon' - (\mathrm{i}k/M_0 + a_{31})\varepsilon = 0. \tag{29}$$

The solution of this equation, say, ε_1. can be written as

$$\varepsilon_1(x) = A_1 \varepsilon_1(0) \exp\left(\int_0^x \mathrm{i}k \, \mathrm{d}x/M_0 \right), \tag{30}$$

where the amplification factor A_1 is given by $A_1 = c_0^2(x)/c_0^2(0)$. This is a wave, subsequently calle the "error wave" or, briefly, the ε-wave, which travels in the direction of the mean flow with the velocity of the mean flow. It will exist along the duct if any imbalance exists at the origin between p and $c_0^2 \rho$. In the case of a subsonic compressible flow of a perfect gas, the speed of sound increases along a divergent duct and decreases along a convergent duct. Therefore, it follows that the ε-wave will be amplified as it travels along a subsonic divergent duct and it will be attentuated along a subsonic convergent duct. Equation (29) is strictly valid for a subsonic low Mach number duct; the decoupling of equation (24) is not justified for subsonic mean flows with a relatively large Mach number and the ε-wave should then be determined by solving equation (24). A numerical method which can be used for the computation of such a three-port acoustic model of a non-uniform duct is described in the Appendix.

This analysis shows that Miles' formulation, equation (6), is approximately valid for subsonic low Mach number ducts with the energy equation implemented in the form of equation (23). The isentropic relationship $p = c_0^2 \rho$ is satisfied with an error given by the ε-wave; however, in the case of a low Mach number duct, the acoustic pressure and particle velocity fluctuations are not sensitive to this wave. Hence, under the conditions stated, Miles [2] is right when he states that he is doing acoustics, but is Gupta's [1] formulation non-acoustical? This question will be taken up in the next section.

2.3. ISENTROPIC PROPAGATION III

Miles' formulation reduces to that given by Gupta [1] if, for the derivative of equation (20), one uses the mathematically proper expression $p' = (c_0^2\rho)'$ in place of the pseudo-derivative $p' = c_0^2\rho'$. The use of the latter, as shown in the previous section, provides an *ad hoc* implementation of equation (23). On these premises, the use of the mathematically proper derivative of equation (20) may be considered non-acoustical because of the imbalance it will cause, when used with equation (20), in the more correct condition for the isentropic propagation, equation (23). Indeed, in the formulation given by Gupta [1], equation (20) is implemented implicitly with its mathematically proper derivative. Equation (23) is hence violated; however, there is a subtle point here—although equation (23) is violated, another condition, which is at least as accurate as equation (23), is implemented as the condition of isentropic propagation. Again, it may be in place to mention here that nowhere in reference [1] is there any reference to the condition for isentropic propagation, equation (34), that will be derived in this section. Therefore, the premises on which the formulation of reference [1] will be justified in this section represents only the present author's own views.

An expression for the entropy fluctuations can be derived also by noting that, in one dimension, the continuity equation for the total quantities is given by

$$D\rho^* + \rho^* v^{*\prime} + \rho^* v^* (\ln S)' = 0, \tag{31}$$

which is the parent form of equation (9). Upon using this in equation (21), one obtains

$$(p^*/c_v^*) Ds^* = Dp^* + \gamma^* p^* (v^{*\prime} + v^*(\ln S)'), \tag{32}$$

where $\gamma^* = c_p^*/c_v^* = \gamma_0 + \gamma$. This relationship can be linearized as usual by assuming that the fluctuations in the ratio of specific heat coefficients, γ, is small to the second order:

$$(p_0^* c_v^*) D_0 s = D_0 p + \gamma_0 p_0 v' + \gamma_0 v_0' p + \gamma_0 (p_0 v + p v_0)(\ln S)'. \tag{33}$$

Hence, the condition for isentropic propagation, equation (11), can be stated as

$$D_0 p + \gamma_0 p_0 v' + \gamma_0 v_0' p + \gamma_0 (p_0 v + p v_0)(\ln S)' = 0. \tag{34}$$

This result requires that only γ, the fluctuations in the ratio of specific heat coefficients, due to acoustic temperature fluctuations, is small to the second order. It is, therefore, at least as accurate as equation (23) which requires that c, the fluctuations in the speed of sound is small to the second order. In fact, equation (34) appears to be slightly more accurate than equation (23) because, since $c^{*2} = (c_0 + c)^2 = \gamma^* R T^*$, where $c_0^2 = \gamma_0 R T_0$ and $2c_0 c = \gamma R T_0 + \gamma_0 R T$, c to be small to second order requires that both γ and T are small to second order, while γ small to second order only implies that c is proportional to T.

Equation (34) can be used with the continuity and momentum equations, equations (9) and (10), respectively, to form a coupled system of equations for p, v and ρ. By using the variable $\varepsilon = c_0^2\rho - p$, these equations can be expressed, after some algebra, in state-space form as

$$\begin{bmatrix} p' \\ v' \\ \varepsilon' \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & b_{13} \\ c_{21} & c_{22} & b_{23} \\ b_{31} & 0 & b_{33} \end{bmatrix} \begin{bmatrix} p \\ v \\ \varepsilon \end{bmatrix}, \tag{35}$$

where the elements of the first 2×2 block of the system matrix are the same as the elements in the formulation of Gupta [1], equation (1), and the remaining elements of the system matrix are given by

$$b_{13} = M_0^2(\ln S)'/(1 - M_0^2)^2, \qquad b_{23} = -M_0^3(\ln S)'/\rho_0 c_0(1 - M_0^2)^2, \qquad b_{31} = (\ln \gamma_0)', \tag{36–38}$$

$$b_{33} = \mathrm{i}k/M_0 + \gamma_0 M_0^2(\ln S)'/(1 - M_0^2) + b_{31}. \tag{39}$$

The off-diagonal element b_{31} is in general small because the temperature dependence of the ratio of the specific heat coefficients is weak for most perfect gases. In the appendix it is shown that, in the case of a subsonic mean flow with a low Mach number, say, $M_0 < 0{\cdot}4$, the effects of the off-diagonal gradients b_{13}, b_{23} and b_{31} on the solution of equation (35) are negligibly small. Thus, if the mean flow Mach number is assumed to be low, then equation (35) can be expressed as a decoupled set of equations consisting of equation (1) and the following first order differential equation for ε; that is,

$$\varepsilon' - (\mathrm{i}k/M_0 + \gamma_0 M_0^2(\ln S)'/(1 - M_0^2) + b_{31})\varepsilon = 0. \tag{40}$$

The solution of this equation, say, ε_2, can be expressed as

$$\varepsilon_2(x) = A_2\varepsilon_2(0)\exp\left(\int_0^x \mathrm{i}k \; \mathrm{d}x/M_0\right), \tag{41}$$

where the amplification factor A_2 is given by $A_2 = \gamma_0(x)p_0(x)/\gamma_0(0)p_0(0)$. The error wave given by equation (40) which, like the ε-wave of equation (29), travels in the direction of the mean flow with the velocity of the mean flow, exists along the duct if any imbalance exists at the origin between p and $c_0^2\rho$ and is amplified or attenuated along the duct according to whether the duct is diverging or converging. It may be of interest to note that, if the ratio of the specific heat coefficients is assumed to be a constant, then equation (40) decouples for any mean flow Mach number and the amplification factor in equation (41) becomes $A_2 = p_0(x)/p_0(0)$. For this case, equation (41) represents the exact solution of equation (35) for the variable ε. However, irrespective of whether γ_0 is assumed to be a constant or is taken as a function of the ambient temperature, the decoupling, from equation (35), of equation (1) is strictly valid for the case of a subsonic low Mach number duct. If the mean flow Mach number is relatively large, then equation (35) should be solved in full, yielding an acoustical three-port model of the duct, as shown in the Appendix.

Thus, for subsonic low Mach number ducts, Gupta's [1] formulation, equation (1), is as acoustical as Miles' formulation. In this case, the energy equation is implemented in the form of equation (34) and the error in the relationship $p = c_0^2\rho$ is predicted as the ε-wave of equation (41); however, the acoustic fluctuations in the fluid pressure and particle velocity are not sensitive to this wave.

3. CONCLUSIONS

The relationship $p = c_0^2\rho$ is strictly valid for isentropic propagation in a homogeneous uniform duct. Nevertheless, it can be used for an *ad hoc* formulation of sound pressure and particle velocity transmission in a non-uniform duct carrying a subsonic low Mach number mean flow. The acoustic density gradient can then be computed either from $p' = c_0^2\rho'$, or from $p' = (c_0^2\rho)'$. The present analysis has shown that these two possibilities are strictly valid for a subsonic low Mach number mean flow and that they are related to two different but correct relationships that describe the isentropic sound propagation in a non-homogenous and non-uniform duct. With respect to the formulations of the problem based on the latter relationships, the *ad hoc* formulations are associated with an error-wave that describes the propagation of the variable $\varepsilon = c_0^2\rho - p$ along the duct. These waves are modified in amplitude as they travel along the duct, and this feature can be utilized as a basis for the comparison of the two formulations in question, namely equations (1) and (6): a larger attenuation of the ε-wave implies a more rapid approach to the underlying assumed relationship $p = c_0^2\rho$, and a smaller amplification will imply a slower divergence from that relationship. As can be seen from equations (30) and (41), the ratio of the magnification factors of the two error waves is given by $A_2/A_1 = \rho_0(x)/\rho_0(0)$. Then, for a subsonic diverging duct, $A_2/A_1 > 1$, and for a converging subsonic duct $A_2/A_1 < 1$. However, the error waves are attenuated along a convergent subsonic duct and amplified along a divergent subsonic duct. Therefore, it follows that Miles' formulation, equation (6), possesses the more favourable error wave characteristics.

If the mean flow Mach number is not low, say, greater than 0·4, then a three-port formulation of the problem using equation (24) or equation (35) should be undertaken. This will require specification of the acoustic density, or the variable ε, at the origin as a boundary condition as well as the usual boundary conditions of the two-port analysis. The situation is simplified if a non-uniform duct connects to a uniform and homogeneous duct. Then, the origin may be taken at the interface of the two ducts and the extra boundary condition can be specified as $\varepsilon = 0$, unless non-isentropic conditions are prevalent at that interface. In the subsonic low Mach number case, $\varepsilon = 0$ will imply no sensible error wave along the non-uniform duct; in the subsonic high Mach number case, however, an ε-wave will develop along the duct (see the figures given in the Appendix).

REFERENCES

1. V. H. GUPTA 1995 *Journal of Sound and Vibration* **182**, 323–327. A note on three-dimensional effects in segmentation approach for analysis of one-dimensional waves in ducts of varying cross-sectional area.
2. J. H. MILES 1995 *Journal of Sound and Vibration* **182**, 328–335. Author's reply.
3. V. H. GUPTA, V. EASWARAN and M. L. MUNJAL 1995 *Journal of Sound and Vibration* **182**, 697–707. A modified segmentation approach for analyzing plane wave propagation in non-uniform ducts with mean flow.
4. J. H. MILES 1981 *Journal of the Acoustical Society of America* **69**(6), 1577–1586. Acoustic transmission matrix of a variable area duct or nozzle carrying a compressible subsonic flow.

APPENDIX: ON THE DECOUPLING OF THE ISENTROPIC DUCT EQUATIONS

This Appendix presents a numerical study on the decoupling of equation (35). The general results of the analysis are applicable also to the decoupling of equation (24). First, a general method is described for solving equation (35) numerically.

Equation (35) can be written in matrix notation as

$$\mathbf{P}'(x) = \mathbf{H}(x)\mathbf{P}(x). \tag{A1}$$

where $\mathbf{P}(x)$ is the vector $\{p(x)\ \ v(x)\varepsilon(x)\}$ and $\mathbf{H}(x)$ is the square state-space matrix. From the theory of the matrizant, the general solution of equation (A1) is given by

$$\mathbf{P}(L) = [\mathbf{Z}]_0^L\mathbf{P}(0), \tag{A2}$$

where $[\mathbf{Z}]_0^L$ denotes the matrizant for the interval $0 \leqslant x \leqslant L$. This is a square 3×3 matrix, which can be evaluated numerically by dividing the interval $0 \leqslant x \leqslant 1$ into N parts by introducing intermediate points $x_1, x_2, \ldots, x_{N-1}$. For simplicity, the lengths of the parts are assumed to be all equal; that is, $l = x_k - x_{k-1}$, $k = 1, 2, \ldots, N$ and $x_N = L = Nl$. Then from the properties of the matrizant,

$$[\mathbf{Z}]_0^L = [\mathbf{Z}]_{x_{N-1}}^{x_N} \cdots [\mathbf{Z}]_{x_1}^{x_2} [\mathbf{Z}]_0^{x_1}, \tag{A3}$$

where $[\mathbf{Z}]_{x_{k-1}}^{x_k}$ denotes the matrizant for the interval $x_{k-1} \leqslant x \leqslant x_k$. If l is small enough, then the state-space matrix, $\mathbf{H}(x)$, can be assumed to be constant in this interval, say, $\mathbf{H}(x) \simeq \mathbf{H}(\xi_k) = \mathbf{H}_k$, where $x_{k-1} \leqslant \xi_k \leqslant x_k$, and the matrizant for this interval can be evaluated approximately from

$$[\mathbf{Z}]_{x_{k-1}}^{x_k} = \exp(\mathbf{H}_k l) = \Phi_k^{-1}\, e^{\Lambda_k l} \Phi_k, \tag{A4}$$

where Φ_k and Λ_k denote the eigenvector and eigenvalue matrices of the 3×3 matrix $\mathbf{H}_k$ and, in this paper, $\xi_k = (x_{k-1} + x_k)/2$. As the number of parts is increased, equation (A3) will converge to the exact solution. Convergence is in general very fast and only a few number of divisions are required for obtaining a satisfactory accuracy.

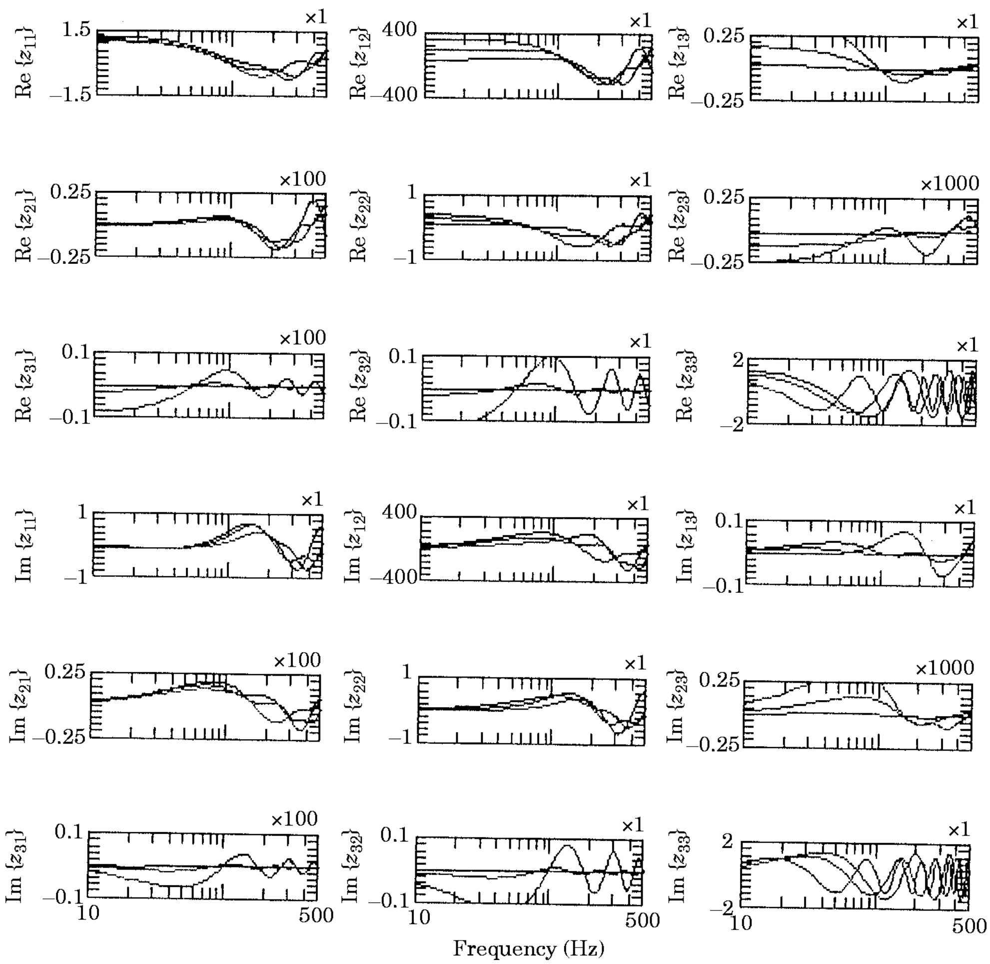

Figure A1. Elements of the matrizant $[\mathbf{Z}]_0^L$ for a subsonic divergent exponential duct with inlet and outlet cross-sectional areas of 0·01 m^2 and 0·02 m^2, a stagnation temperature of 295 K, a stagnation pressure of 101·3 Pa and inlet Mach numbers $M_0(0) = 0{\cdot}3$, 0·593 and 0·9. The larger the Mach number is, the greater is the apparent deviation of the off-diagonal elements Z_{13}, Z_{23}, Z_{31} and Z_{32}, from the zero line.

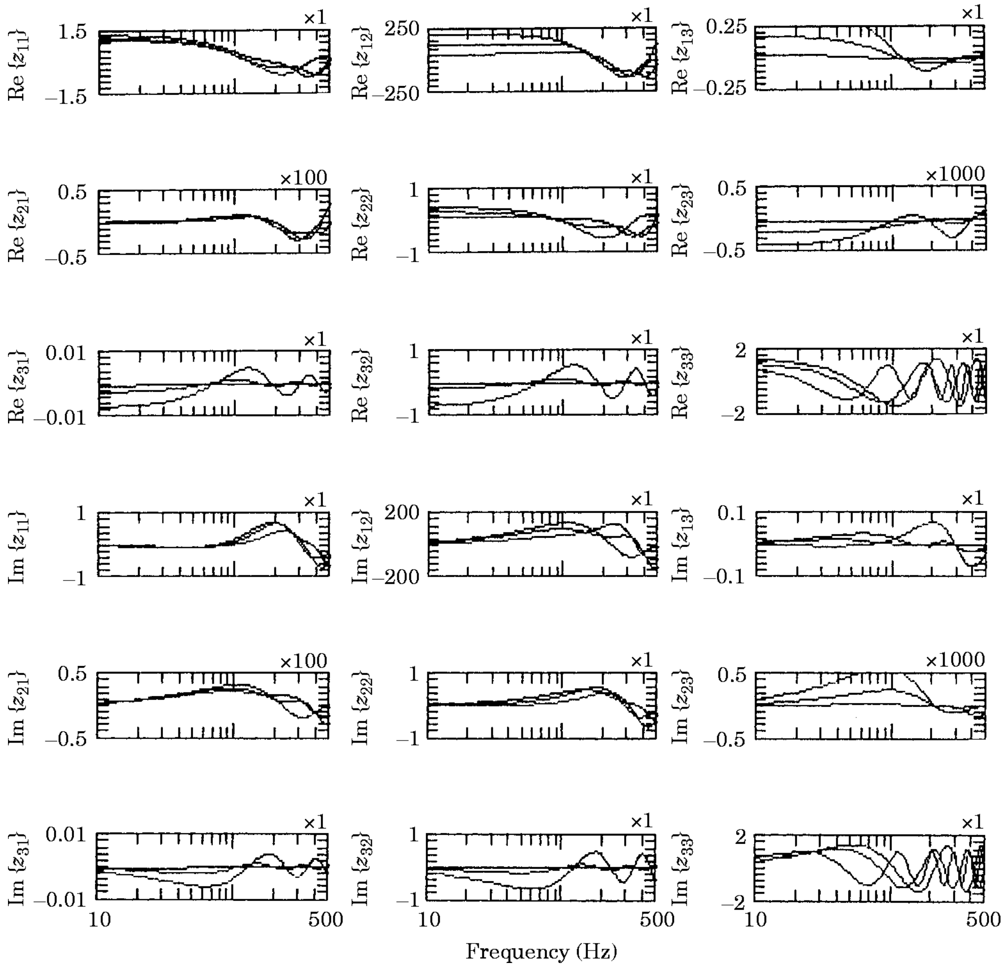

Figure A2. Elements of the matrizant $[\mathbf{Z}]_0^L$ for the same duct of Figure A1 but with a stagnation temperature of 600 K, a stagnation pressure of 121·2 Pa and inlet Mach numbers $M_0(0) = 0{\cdot}3$, 0·6 and 0·9. The larger the Mach number is, the greater is the apparent deviation of the off-diagonal elements, Z_{13}, Z_{23}, Z_{31} and Z_{32}, from the zero line.

As can be seen from equation (A3) and (A4), if b_{31} and b_{13} in equation (35) were zero, then the off-diagonal elements on the third row and the third column of the matrizant $[\mathbf{Z}]_0^L$ would also be zero. Conversely, equation (35) can be considered as being decouplable if those off-diagonal elements of $[\mathbf{Z}]_0^L$ are all approximately equal to zero.

The variation of the real and imaginary parts of the elements of the matrizant $[\mathbf{Z}]_0^L$, say, Z_{ij}, $i, j = 1, 2, 3$, with frequency and inlet Mach number are shown in Figure A1 for the diverging exponential duct considered by Miles [4] and also by Gupta *et al*. [3]. The inlet and outlet cross-sectional areas of this duct are 0·01 m^2 and 0·02 m^2, respectively, the stagnation temperature is 295 K and the stagnation pressure is 101·3 kPa. The ambient gas is assumed to be dry air and the temperature dependence of the specific heat coefficients is taken into account in the computations. The elements of the matrizant are plotted for inlet mean flow Mach numbers of $M_0(0) = 0{\cdot}3$, 0·593 and 0·9. The intermediate value corresponds to a mass flow rate of 2 kg/s which is the case considered in references [3, 4]. For completeness, all the elements of the matrizant $[\mathbf{Z}]_0^L$ are shown in Figure A1. The main interest here, however, is on the off-diagonal elements Z_{13}, Z_{23}, Z_{31} and Z_{32}. The elements in the first 2×2 block of $[\mathbf{Z}]_0^L$, Z_{11}, Z_{21}, Z_{12} and Z_{22}, correspond to equation (1) and their magnitudes for this duct are given also in reference [3] for the 2 kg/s mass flow rate case.

The mean flow Mach numbers are not marked on the curves in Figure A1 for clarity. For the elements of the matrizant that are of interest here, the curves can be matched with the three inlet Mach number considered by noting the following trends. The Z_{33} characteristics are slightly damped sinusoids, the period of which on the frequency axis is approximately proportional to the inlet Mach number; and the larger is the inlet Mach number, the greater is the apparent deviation of the off-diagonal elements Z_{13}, Z_{23}, Z_{31} and Z_{32} from the zero line. The latter characteristics are also attenuated slightly with frequency.

As can be seen from Figure A1, for inlet Mach numbers less than or equal to 0·3, the off-diagonal elements Z_{13}, Z_{23}, Z_{31} and Z_{32} may be assumed to be approximately equal to zero; however, as the inlet Mach number increases they gain prominence and the decoupling of equation (35) may no longer be justified. These characteristics are somewhat insensitive to changes in the outlet cross-sectional area. In Figure A2 are shown, for the same duct, the elements of the matrizant for a stagnation temperature of 600 K and a stagnation pressure of 121·2 kPa, for inlet Mach numbers of 0·3, 0·6 and 0·9, which can be matched with the characteristics as described above for Figure A1. It is seen that increasing the stagnation temperature and pressure has the effect of increasing the off-diagonal elements Z_{13}, Z_{23}, Z_{31} and Z_{32} for the same inlet Mach number and, consequently, lowering of the threshold value of the mean flow Mach number for which the decoupling of equation (35) will be valid.

Thus, it appears that, equation (35) can be decoupled to equation (1) and equation (40) if the mean flow Mach number is less than the limit indicated above, that is, less than about 0·3 or 0·4. Although a similar numerical study has not been carried out for equation (24), this conclusion should be valid also for the decoupling of this equation because, for the low mean flow Mach number case, the difference between the two formulations pertains to only small terms.